A partially backlogged two-warehouse EOQ model with non-instantaneous deteriorating items, price and time dependent demand and preservation technology using interval number

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Abstract: In this paper, a two-warehouse EOQ model for non-instantaneous deteriorating item with price and time dependent demand is developed under imprecise environment. To reduce the loss due to deterioration, the concept of preservation technology investment is incorporated. In order to make the model more general, shortages are allowed and are partially backlogged. Further, the impreciseness of the parameters like coefficients of demand rate, holding costs, deterioration costs, shortage cost and lost sale cost are assumed as interval numbers which are made crisp by parametric functional form representation of interval numbers. The objective of this model is to find an optimal ordering policy with a view to minimise total average cost. To illustrate the developed model, a numerical example is taken in its support and also sensitivity analysis is carried out with respect to some important parameters.

Keywords: two-warehouse; non-instantaneous deteriorating item; preservation; partial backlogging; interval number; parametric functional form.

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1 Introduction

In today's world, the word inventory problem has become a household term. Inventory problem arises in every sphere of life right from one's kitchen to any business enterprise or industry. In this present era of high competition, every business organisation wants to be on the top of the market, and is focused to maintain a good relationship with the customers, for that a perfect inventory policy is required. This unfolds a research area to many academicians and practitioners in the field of inventory management. A perfect inventory control system always represents a real market situation and keeps all the parameters like demand, storage space, etc. up to mark. Since the inception of stocking process, the researchers have been developing various inventory models incorporating different parameters like demand, deterioration, shortage, etc. representing the real life scenarios so as to get insights in decision making. Among these developed models, enormous emphasis has been given to develop models for deteriorating items in the recent times. Panda et al. (2017) proposed a volume flexible deteriorating inventory model with price sensitive demand to maximise total profit per cycle using real-coded genetic algorithm (RCGA). Kumar (2019) developed an inventory planning problem for deteriorating items with time varying demand and parabolic time dependent holding cost. In the aforesaid model, the salvage value was incorporated and it was found that the salvage coefficient has least impact on the parameters of the inventory system. Jaggi et al. (2019a) also studied a two-echelon supply chain inventory model for deteriorating items with displayed stock dependent demand. A partially backlogged inventory model with linear rate of deterioration and selling price dependent demand was developed by Sahoo et al. (2019). The said model was solved analytically to minimise total inventory cost. The work of Jaggi et al. (2019b) is also worth mentioning in the field of inventory management for deteriorating items.

In most of the inventory models, it is assumed that items get deteriorated instantly on their arrival in the system. This type of deteriorating item is termed as instantaneous deteriorating item. But, there are large numbers of items, as for example, electronic goods, bloods, milk, fresh fruits, vegetables, etc. that maintain their quality or original condition for a definite period of time. During that span, no deterioration comes into picture. These items may be classified as non-instantaneous deteriorating items. Consideration of such type of items in inventory models is much realistic, and this has been an object for study in the recent decade. Though these items do not deteriorate for a fixed period of time, but after that definite span of time the effect of deterioration, however, cannot be overlooked for long run of a business setup. To shrink the deteriorating rate, many business enterprises have considered preservation technologies in inventory system in order to reduce economic loss due to deterioration. After the pioneering contribution by Hsu et al. (2010), lots of models have been developed with the use of preservation technology. The works by Sarkar et al. (2017) and Zhang et al. (2016) are worth mentioning in this regard. For non-instantaneous deteriorating items, Dye (2013) was first to incorporate preservation technology to develop the model with constant demand and time dependent deterioration rate. Recently, Li et al. (2019) developed a model for non-instantaneous items with price dependent demand under the use of preservation technology investment. They extended the Dye's (2013) model to study interesting fact that preservation investment affects not only the deterioration rate but also the length of no-deterioration period of the item.

Table 1 Comparison of inventory models with single or two-warehouse under preservation technology related to this present article

Authors	*Demand pattern*	*Deterioration*	*Warehouse*	*Shortages*	*Inventory costs/parameters*
Hsu et al. (2010)	Constant	Instantaneous	Single	Yes	Crisp
Dye and Hsieh (2012)	Constant	Instantaneous	Single	Yes	Crisp
Lee and Dye (2012)	Stock dependent (linear)	Instantaneous	Single	Yes	Crisp
Dye (2013)	Constant	Non-instantaneous	Single	Yes	Crisp
He and Huang (2013)	Price dependent	Instantaneous	Single	No	Crisp
Hsieh and Dye (2013)	Time dependent	Instantaneous	Single	No	Crisp
Singh et al. (2013)	Constant	Instantaneous	Two	No	Crisp
Mishra (2014)	Time dependent (linear)	Non-instantaneous	Single	Yes	Crisp
Singh and Rathore (2015)	Time dependent	Instantaneous	Single	Yes	Crisp
Tsao (2016)	Stochastic (Poisson process)	Non-instantaneous	Single	No	Crisp
Singh and Rathore (2016)	Stock dependent	Instantaneous	Two	Yes	Crisp
Mishra et al. (2017)	Price and stock dependent	Instantaneous	Single	Yes	Crisp
Pal et al. (2018)	Constant	Non-instantaneous (random variable)	Single	Yes	Crisp
Shah and Naik (2018)	Price and time dependent	Instantaneous	Single	Yes	Crisp
Rathore and Singh (2018)	Advertisement and price dependent	Instantaneous	Two	Yes	Crisp
Rathore et al. (2018)	Advertisement, price and time dependent	Instantaneous	Two	No	Crisp
Bardhan et al. (2019)	Stock dependent	Non-instantaneous	Single	No	Crisp
Pervin et al. (2019)	Price and time dependent	Instantaneous	Single	Yes	Crisp
This paper	Price and time dependent (linear)	Non-instantaneous	Two	Yes	Imprecise

Most of inventory models are developed with different realistic assumptions, and then a suitable optimisation problem is formulated as cost minimisation or profit maximisation problem. To solve these types of optimisation problems in an uncertain situation, researches employ different approaches like, stochastic approach, fuzzy approach, fuzzy-stochastic approach and interval approach. Among these approaches, interval approach is more significant. In recent times, it has received considerable attention from modelling community. In interval approach, we represent impreciseness of the parameters by interval numbers. For details, one may refer to Rahman et al. (2020a) and Ruidas et al. (2019).

In the next section, literature review has been given to get a current state of knowledge about different inventory models in the context of price and time dependent demand, preservation technology and interval approach for deteriorating items.

2 Literature review

The study of inventory models started in the year 1915. Harris (1915) was the first to consider economic order quantity (EOQ) inventory model considering constant rate of demand and no deterioration. But in reality, it is quite natural that most of the items to be stored deteriorate or decay over time. Thus, in order to make a realistic inventory model the effect of deterioration cannot be neglected. In literature, deterioration is defined, in general, as the decay, damage, expiration, spoilage, obsolescence, loss of quality or value of the stored items. Ghare and Schrader (1963) was the first to consider EOQ model for deteriorating items. They proposed an EOQ model for exponentially decaying items with constant demand. Since then, lots of models have been developed for deteriorating items. Covert and Philip (1973) presented an inventory model for deteriorating items where deterioration rate follows Weibull distribution. Later, Deb and Chaudhuri (1986) developed a model for deteriorating items with time dependent demand. Bhunia et al. (2009), Dave and Patel (1981), Sarkar and Sarkar (2013) and Wee (1997) are some significant contributions in this direction. For detailed survey, the works of Bakker et al. (2012) and Goyal and Giri (2001) may be referred. Recently, Singh et al. (2018) proposed an EOQ model with ramp type demand for decaying items. They considered items deteriorate at the rate following three-parameter Weibull distribution. Khurana et al. (2018) developed a production inventory model for time dependent demand and deterioration. To make it a general one, they allowed production rate to depend on market demand. Khurana and Chaudhary (2018) proposed an inventory model where demand is time and stock dependent. They studied the model under partial backlogging (both constant and time dependent) to see the effect on total cost. Tiwari et al. (2018) proposed a joint pricing and inventory model under two-level partial trade credit with partial backlogging shortages. They solved the model analytically and designed a solution algorithm to maximise profit. Although most of the inventory models assume that items start deteriorating instantly from the time of arrival in the stock, Wu et al. (2006) was the first to argue that there are items whose deterioration occur after a definite period of time. In the recent decade, group of researchers have studied inventory policy for non-instantaneous deteriorating items, for instance, Chung (2009), Geetha and Uthayakumar (2010), Ouyang et al. (2006) and Rabbani et al. (2015) are some worth mentioning papers in this regard. Tiwari et al. (2017) also developed a model for non-instantaneous deteriorating items with stock dependent demand under inflation and partial backlogging. They applied particle swarm optimisation to get the optimal solution. Recently, Bhaula et al. (2019) studied a model for non-instantaneous deteriorating items under permissible delay in payment. They allowed successive price discount in selling price to increase selling of items during the replenishment cycle which is generally observed in present competitive market.

The classical inventory models are mainly developed by assuming that inventory manager has single storage facility called own warehouse (OW) with infinite capacity. But thinking in more practical terms, when suppliers offering concession on bulk purchases or high demand of items or seasonality, inventory manager may decide to purchase goods in large quantity at a time. To store excess of the items purchased, we need another storing facility. This extra storage facility is generally taken on hire basis called rented warehouse (RW), which is assumed to hold goods abundantly. The effect of two-warehouse on inventory policy was first discussed by Hartley (1976). He developed a model with constant demand without allowing shortages. Sarma (1987) further modified his model allowing fully backlogged shortages. Pakkala and Achary (1992) proposed a production inventory model for deteriorating items with constant demand. Later, Bhunia and Maiti (1998) and Lee and Ying (2000) developed two-warehouse deteriorating inventory models with time dependent demand. Yang (2006) proposed a two-warehouse inventory model with constant demand and inflation under partial backlogging. Bhunia et al. (2015), Gayen and Pal (2009) and Shabani et al. (2016) also considered deterioration from the time of arrival of goods in their developed two-warehouse inventory models. However, Jaggi et al. (2015) developed two-warehouse model for non-instantaneous deteriorating items assuming constant demand and complete backlogging under FIFO policy. Palanivel and Uthayakumar (2016) proposed a two-warehouse model for non-instantaneous deteriorating items with constant demand and inflation over finite time horizon and partial backlogging. Palanivel et al. (2016) further extended the work of Palanivel and Uthayakumar (2016) considering stock dependent demand. Tiwari et al. (2016) proposed a two-warehouse model for non-instantaneous deteriorating items under trade credit policy. Udayakumar and Geetha (2018) also studied a two-warehouse model for non-instantaneous deteriorating items under trade credit policy. Shaikh et al. (2019a) considered a two-warehouse inventory model for non-instantaneous deteriorating items under inflationary conditions with stock dependent demand.

In all the inventory models discussed above, the deterioration rate is either constant or variable, which is not subject to control. But this deterioration may be put under control and reduced by applying a suitable preservation technology. For instance, refrigeration equipment is used in supermarket to reduce the deterioration rate of fruits, vegetables and sea-foods. Drying, cooling, heating, vacuum packing, etc. are some examples of the preservation techniques. So far as our survey is concerned, Hsu et al. (2010) were probably the first to develop a single storage inventory model for deteriorating (instantaneous) items using preservation technology allowing inventory manager to invest a certain amount per cycle to reduce deterioration rate. They considered demand to be constant and backlogging rate is a linear function of waiting time. Dye and Hsieh (2012) extended Hsu et al.'s (2010) model by assuming time-varying deterioration rate and preservation technology cost as a function of replenishment cycle length. Hsieh and Dye (2013) proposed a single storage production inventory model under preservation technology with time dependent demand. Dye (2013) studied the effect of preservation technology on non-instantaneous deteriorating items under partial backlogging with constant demand rate and preservation cost as a function of replenishment cycle length. He and Huang (2013) developed a single storage inventory model for seasonable deteriorating products with price dependent demand under the use of preservation technology. Mishra (2014) also studied non-instantaneous deteriorating items under the use of preservation technology. The author assumed demand is a linear function of time, and allowed partial backlogging during shortage period. Singh and Rathore (2015) developed a single storage optimal policy for deteriorating items under preservation technology, constant inflation, time dependent demand and trade credit. Tsao (2016) considered a model for non-instantaneous deteriorating items with stochastic demand under the use of preservation technology. Mishra et al. (2017) developed a single storage inventory model for deteriorating item over finite time horizon. The authors considered demand as price and stock dependent, and studied the model under both partial and complete backlogging. Shah et al. (2018) studied the effect of preservation technology investment on customer service level. They considered demand to depend on time, selling price, service level, and derived analytical form of dynamic service investment using Pontryagin's maximum principle. Pal et al. (2018) have proposed an optimal inventory policy with constant demand for non-instantaneous deteriorating items. In their model, the authors considered deterioration start time as random variable and preservation technology is used to reduce the deterioration rate. Bardhan et al. (2019) also used preservation technology on non-instantaneous deteriorating items to develop a single storage inventory model with stock dependent demand. Recently, Singh (2019) proposed a production model to study the effect of preservation technology on total cost. The model was solved analytically with and without shortages, and found to be suitable for items with finite shelf-life.

Demand of an item is also a major concerning point for an inventory manager. In literature, we have seen different types of demand to exist. Demand may be constant or function of other variables like time, stock, selling price, advertisement cost, etc. In today's competitive world, apart from quality, selling price of an item plays an important role in customer's demand. It is one of the decisive factors to the customers for purchasing an item. It is seen that almost all products are price sensitive that is, increase in selling price decreases demand and decrease in selling price increases demand. Also, demand is seen to vary with time. Thus, it is quite practical to consider demand to depend jointly on time and selling price of the item. In the recent years, price and time dependent demand has drawn considerable attention of researchers. Chang et al. (2006), Farughi et al. (2014), Maihami and Kamalabadi (2012), Valliathal and Uthayakumar (2011) and You (2005) are some notable contributions related to price and time dependent demand. Recently, Saha and Sen (2019) have developed an EOQ model for deteriorating items with price and time dependent demand, negative exponential inflation rate and partial backlogging. Shah and Naik (2018) developed a single warehouse EOQ model for deteriorating items with price and time dependent demand under full advance payment with or without shortages. They studied the model under preservation technology investment to see its effect on average total profit of the system. Rathore et al. (2018) have proposed a two-warehouse inventory model with advertisement, time and price dependent demand under the use of preservation technology. To authors' knowledge, a few research papers have been reported in literature in connection with price and time dependent demand, preservation technology investment for two-warehouse inventory system. A comparison of models with single or two-warehouse for instantaneous and non-instantaneous deteriorating items under preservation technology investment has been presented in Table 1.

While formulating mathematical model for decision making, it is generally assumed that demand, deterioration and inventory costs like holding cost, ordering cost, deterioration cost, etc. are precise or constant. However, in reality, we often come across the situations, especially in engineering design problems, management problems, where it is difficult to get exact value of the parameters. This may be due to human error or lack of sufficient data or fluctuation in data or unexpected situations. So, in order to deal with such impreciseness or uncertainty in parameters, generally, researchers use stochastic approach or fuzzy approach or fuzzy-stochastic approach or interval approach. In stochastic approach, parameters are assumed as random variables with known probability distribution whereas in fuzzy approach parameters are assumed as fuzzy sets with suitable membership function. Roy et al. (2009) developed an inventory model with stock dependent demand and random planning horizon. Bhunia and Shaikh (2011) used Weibull distribution for deterioration in their paper. Duari and Chakrabarti (2016) also developed two-warehouse inventory model with Weibull distribution for deterioration and considered stock dependent demand. Saha and Sen (2017) proposed an inventory model where deterioration follows uniform, beta and triangular distributions. But, it is not always possible to estimate a probability distribution function due to lake of historical data, as for example, the case of newly launched products. That is why many researchers have developed models in fuzzy environment. Chang et al. (1998) proposed an economic reordering point model for fuzzy backordered quantity. After that, Yao et al. (2000) developed a fuzzy EOQ inventory model with fuzzy order quantity and fuzzy total demand quantity. Sen et al. (2016) discussed a fuzzy inventory model considering triangular fuzzy numbers. Very recently, Indrajitsingha et al. (2019) developed a two-warehouse inventory model with fuzzy approach, where holding cost, deterioration rate, shortage cost and lost sale cost are considered as triangular fuzzy numbers. However, it is not always an easy task to select a suitable membership function. Therefore, in such cases, interval numbers may come out to be effective because of its simplicity in representing impreciseness of parameters. For this reason, interval number approach is growing interest among the researchers over the recent years. In the next paragraph, a discussion on some important interval related works is done.

Gupta et al. (2007) developed an inventory model under interval uncertainty with demand depending on selling price, display stock level and frequency of advertisement. They used genetic algorithm (GA) to obtain the optimum solution. Gupta et al. (2009) further developed a model with uniform demand where cost parameters are taken as interval number. They proposed a RCGA with ranking selection to obtain the optimal solution. Chakrabortty et al. (2010) used multisection technique and interval order relation to solve a purchasing inventory model. They considered holding cost, ordering cost and demand as interval numbers. Chakrabortty et al. (2013) also proposed an EOQ model under interval uncertainty. They considered inventory costs, ordering quantity and demand as interval numbers. They proposed an optimisation technique based on division criteria of prescribed/accepted search region. Bhunia and Shaikh (2016) proposed a two-warehouse inventory model under inflation and linear time dependent demand. They assumed interval valued inventory cost and used particle swarm optimisation technique to obtain the optimal solution. A partially integrated production-inventory model was developed by Bhunia et al. (2017) with selling price and marketing cost dependent demand, and interval valued inventory cost. They used PSO-CO to solve the problem.

Ruidas et al. (2018) also developed an EOQ model for defective items and used PSO-CO to solve the model. Shaikh et al. (2019b) proposed a two-warehouse model with advance payment scheme, price dependent demand and interval valued cost parameters. They obtained and compared solution by three variants of PSO (PSC-CO, WQPSO and GQPSO). Mondal et al. (2019) developed an EOQ model for ameliorating items following three-parameter Weibull distributed amelioration and deterioration. They considered advertisement and time dependent demand, and interval valued inventory cost parameters. They studied the model under both crisp and interval environment. They used three variants of QPSO to solve the model in both the environments. Recently, Rahman et al. (2020b) introduced a new type of interval number called type-2 interval number. To increase the flexibility, they considered the boundary of an interval as interval number. They extended the earlier definition and proposed new definition of interval arithmetic and order relation for type-2 interval numbers. Further, they applied this concept to solve classical EOQ model.

As we know, in interval approach imprecise parameters are generally represented by interval numbers. As a result, two separate objective functions corresponding lower and upper limit of an interval are formulated, which leads to multi-objective formulation. Thus, existing classical or iterative methods cannot used to get optimal solution. For this, heuristic or meta-heuristic search methods are generally employed to obtain optimal solution with the help of interval arithmetic and interval order relation (Karmakar and Bhunia, 2012). Instead of formulating two separate objective functions, parametric functional form representation of interval number may be used to convert interval valued parameters so that we get a closed form of expression of the objective function, which can be solved by existing classical or iterative techniques. This concept of parametric functional form has already been used in Mahapatra and Mandal (2012), Pal et al. (2013) and Pal and Mahapatra (2016, 2017). In the field of inventory control theory, for the first time, this concept was incorporated by Das and Roy (2018). They developed a single storage EOQ model for non-instantaneous deteriorating items considering inventory costs as interval numbers. They used parametric functional form to remove the interval uncertainty of the parameters.

3 Research gap and objective

In the existing literature of inventory modelling, lots of research models have been developed in crisp and fuzzy environments and some in stochastic environment. In the recent decade, some significant contributions have also been made in inventory control theory for instantaneous deteriorating items in the interval environment. But, a few have been developed for non-instantaneous deteriorating items (Das and Roy, 2018; Shaikh et al., 2019a). So, motivated by the works of Das and Roy (2018), Rathore et al. (2018), Shah and Naik (2018) and based on the above literature review, we have observed that no one has studied two-warehouse inventory model for non-instantaneous deteriorating items in the context of price and time dependent demand with preservation technology, when inventory cost parameters and coefficients of demand are imprecise (interval uncertainty) in nature. Thus, in this paper, an attempt has been made to fill up this gap. The developed model is a two-warehouse EOQ model for non-instantaneous deteriorating items that embodies the following features:

1. Demand is selling price and time dependent.
2. Preservation technology investment is incorporated to reduce deterioration rate.
3. Shortages are allowed to occur and are partially backlogged. The rate of backlogging depends upon waiting time to the next replenishment.
4. Coefficients of demand function and all the inventory cost parameters: holding costs, deterioration costs, shortage cost and lost sale cost are constant but imprecise. The impreciseness of the parameters is presented in terms of interval number.
5. Parametric functional form representation of interval is used to remove impreciseness of the parameters.

Based on these above features, the corresponding total average cost function is formulated. The objective of this research work is to determine an optimal ordering policy that is to find optimal ordering quantity, optimal replenishment time, optimal pricing and optimal preservation technology investment simultaneously so that the present worth of the total average cost of the system is minimised. The proposed model will be useful to the retailers or decision makers because it will help them in taking important replenishment decisions.

The rest of the paper is organised as follows: in Section 4, a detailed description of assumptions and notations used in this paper is provided. In Section 5, mathematical model is formulated and total average cost is calculated. In Section 6, solution procedure is discussed. In Section 7, a numerical example is taken to illustrate the model. In Section 8, sensitivity analysis is carried out to study the effects of changes in different parameters of the system. Finally, in Section 9, conclusions are drawn and future research direction is indicated.

4 Assumptions and notations

To develop the model, the following assumptions are made.

4.1 Assumptions

- The model deals with single non-instantaneous type of deteriorating item over a period.
- Demand is selling price and time dependent, where the coefficients of the demand functions are constant but imprecise.
- Planning horizon is infinite.
- Lead time is zero.
- Replenishment rate is infinite.
- The system involves a two-warehouse system, one is OW with limited capacity and other is RW with infinite capacity. For economic reasons, items are first consumed from RW, and then from OW.

- Deterioration comes into play after a certain period of time, and the rates of deterioration are different for warehouses.
- The holding costs per time per unit, deterioration costs per unit, shortage cost per time per unit and lost sale cost per time per unit are assumed to be constant but imprecise.
- Holding cost at RW is higher than that of OW due to better preserving facility at RW. OW is situated at the heart of the market and RW is little away from the market place.
- Transportation cost and transportation time are negligible.
- There is no repair or replacement of the items.
- Shortages are allowed and are partially backlogged. The backlogging rate is dependent on the length of waiting time for next replenishment.
- Preservation technology is used in order to control the loss due to deterioration.

4.2 Notations

The following notations are used throughout this paper.

4.2.1 Decision variables

p per unit selling price of the item

t_r time at which inventory level in RW reaches to zero

t_0 time at which inventory level in OW reaches to zero

T cycle length

ξ preservation technology cost per unit time.

4.2.2 Parameters

$\hat{D}(p, t)$	demand rate of the form $\hat{a} - \hat{b}p + \hat{c}t$ where $\hat{a}, \hat{b}$ and $\hat{c}$ are constants but imprecise
$B(t) = \dfrac{1}{1+\delta(T-t)}$	backlogging rate where $\delta \in (0, 1)$ is the backlogging parameter and $T - t$ is the waiting time $t_0 \le t \le T$
W	capacity of OW
t_d	time after which deterioration of the item starts
S_T	maximum amount of demand backlogged per cycle
Q	quantity to be ordered at the beginning of the cycle
$Q_0(t)$	OW inventory level at any time $t \in [0, t_0]$

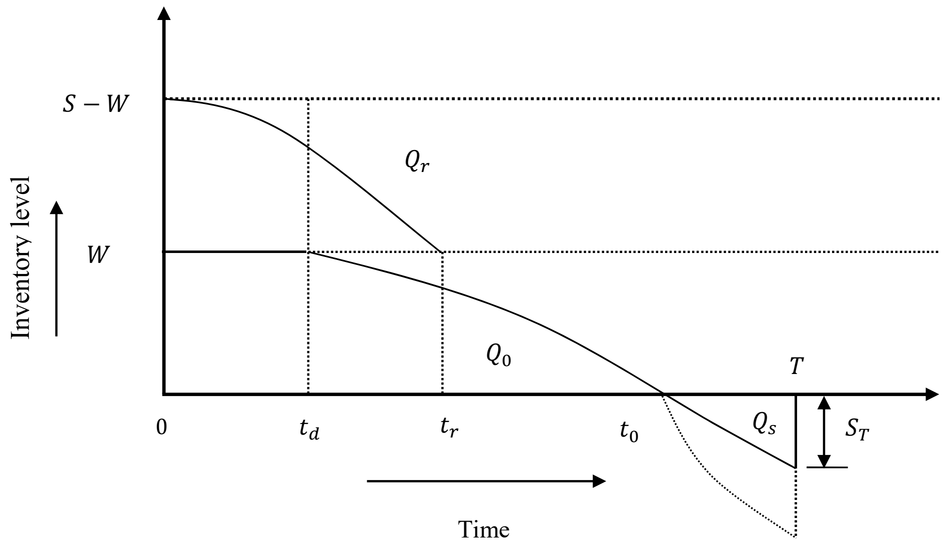
$Q_r(t)$	RW inventory level at any time $t \in [0, t_r]$
$Q_s(t)$	inventory level at any time $t \in [0, t_0]$
$m(\xi) = 1 - e^{-\gamma\xi}$	portion of reduced deterioration rate where $\gamma > 0$
θ_{RW}	deterioration rate at RW, $0 < \theta_{RW} << 1$
θ_{OW}	deterioration rate at OW, $0 < \theta_{OW} << 1$, $\theta_{OW} > \theta_{RW}$
$\tau_i = \theta_i(1 - m(\xi))$	resultant deterioration rate, $i = RW, OW$
A	ordering cost per order
x	per unit purchasing cost of the item, $p > x$
$\hat{H}_{RW} = [h_1L, h_1R]$	imprecise holding cost per time per item in RW
$\hat{H}_{OW} = [h_2L, h_2R]$	imprecise holding cost per time per item in OW
$\hat{d}_{RW} = [d_1L, d_1R]$	imprecise deterioration cost per item in RW
$\hat{d}_{OW} = [d_2L, d_2R]$	imprecise deterioration cost per item in OW
$\hat{S}_c = [S_cL, S_cR]$	imprecise shortage cost per time per item
$\hat{L}_c = [L_cL, L_cR]$	imprecise lost sale cost per time per item
TAC	total average cost per cycle.

5 Mathematical model formulation

In this section, the mathematical model based on the assumptions and notations made in Section 4 is developed, and total average cost is calculated. The considered model is a two-warehouse inventory model for a single non-instantaneous deteriorating item with partial backlogging. The model is described as follows: at $t = 0$, a lot of size Q units enter the system. A portion S_T of it is used to clear all the backorders. Let $Q = S + S_T$ so that on hand inventory level at the beginning of the cycle is S. Out of these S units, W units are kept in OW, and the rest that is $S - W$ units are stored in the RW. The items of OW will be consumed only after consuming the items stored in RW. Since the deterioration is non-instantaneous, initially, the items do not deteriorate up to the time period t_d and then after the deterioration of items starts. Therefore, in the RW, during the time interval $[0, t_d]$, the inventory level decreases only due to demand, and the inventory level further depletes due to joint effect demand and deterioration during the time interval $[t_d, t_r]$. At $t = t_r$, the inventory level in the RW drops to zero. Whereas, in the OW, during the time interval $[0, t_d]$, the inventory level remains unchanged. And during the time interval $[t_d, t_r]$, the inventory level in OW is depleted only due to deterioration. Further, during the time interval $[t_r, t_0]$, the inventory level in OW decreases due to combine effect of demand and deterioration, and finally it drops to zero at $t = t_0$. In the shortage period

$[t_0, T]$, demand is partially backlogged at the rate $\frac{1}{1+\delta(T-t)}$. The behaviour of the inventory model during the period is demonstrated in Figure 1.

Figure 1 Graphical representation of a two ware house inventory model



Since the inventory level in RW decreases during $[0, t_d]$ only due to demand, the differential equation representing the inventory level is given by:

$$\frac{dQ_r}{dt} + \hat{D} = 0, \; 0 \le t \le t_d \tag{1}$$

With the condition $Q_r(0) = S - W$, the solution of equation (1) is

$$Q_r(t) = S - W - \left(\hat{a} - \hat{b}p\right)t - \frac{\hat{c}}{2}t^2, \; 0 \le t \le t_d \tag{2}$$

In the next interval $[t_d, t_r]$, the inventory level in RW decreases due to demand and deterioration. Therefore, the governing differential equation is as below:

$$\frac{dQ_r}{dt} + \tau_{RW} Q_r + \hat{D} = 0, \; t_d \le t \le t_r \tag{3}$$

With the boundary condition $Q_r(t_r) = 0$, the solution of equation (3) is

$$Q_r(t) = -e^{-\tau_{RW}t}\left[\left(\hat{a} - \hat{b}p\right)t + \left\{\tau_{RW}\left(\hat{a} - \hat{b}p\right) + \hat{c}\right\}\frac{t^2}{2} + \frac{\hat{c}\tau_{RW}}{2}t^3\right] + e^{-\tau_{RW}t}A_1, \; t_d \le t \le t_r \tag{4}$$

where

$$A_1 = \left(\hat{a} - \hat{b}p\right)t_r + \left\{\tau_{RW}\left(\hat{a} - \hat{b}p\right) + \hat{c}\right\}\frac{t_r^2}{2} + \frac{\hat{c}\tau_{RW}}{2}t_r^3 \tag{5}$$

Since $Q_r(t)$ is continuous at $t = t_d$, putting $t = t_d$ in equations (2) and (4), the value of maximum inventory level per cycle (S) is calculated as below:

$$\begin{aligned} S = {} & e^{-\tau_{RW}t}\left[\left(\hat{a} - \hat{b}p\right)\left(t_r - t_d\right) + \left\{\tau_{RW}\left(\hat{a} - \hat{b}p\right) + \hat{c}\right\}\frac{\left(t_r^2 - t_d^2\right)}{2} + \frac{\hat{c}\tau_{RW}}{2}\left(t_r^3 - t_d^3\right)\right] + W \\ & + \left(\hat{a} - \hat{b}p\right)t_d + \frac{\hat{c}}{2}t_d^2 \end{aligned} \tag{6}$$

In OW, during the time interval $[0, t_d]$, there is no change in the inventory level, and during $[t_d, t_r]$, the inventory level decreases due deterioration only. Therefore, the corresponding differential equations are:

$$\frac{dQ_0}{dt} = 0,\ 0 \le t \le t_d \tag{7}$$

$$\frac{dQ_0}{dt} + \tau_{OW}Q_0 = 0,\ t_d \le t \le t_r \tag{8}$$

The solutions to equations (7) and (8) with the boundary conditions $Q_0(0) = W$ and $Q_0(t_d) = W$ are given below:

$$Q_0(t) = W,\ 0 \le t \le t_d \tag{9}$$

$$Q_0(t) = We^{\tau_{OW}(t_d - t)},\ t_d \le t \le t_r \tag{10}$$

During $[t_r, t_0]$, the inventory level in OW depletes due to both demand and deterioration. Thus, the rate of change in the inventory level is given by:

$$\frac{dQ_0}{dt} + \tau_{OW}Q_0 + \hat{D} = 0,\ t_r \le t \le t_0 \tag{11}$$

With the boundary condition $Q_0(t_0) = 0$, we obtain the solution to equation (11) as below:

$$Q_0(t) = e^{-\tau_{OW}t}\left[\left(\hat{a} - \hat{b}p\right)t + \left\{\tau_{OW}\left(\hat{a} - \hat{b}p\right) + \hat{c}\right\}\frac{t^2}{2} + \frac{\hat{c}\tau_{OW}}{2}t^3\right] + e^{-\tau_{OW}t}A_2,\ t_r \le t \le t_0 \tag{12}$$

where

$$A_2 = \left(\hat{a} - \hat{b}p\right)t_0 + \left\{\tau_{OW}\left(\hat{a} - \hat{b}p\right) + \hat{c}\right\}\frac{t_0^2}{2} + \frac{\hat{c}\tau_{OW}}{2}t_0^3 \tag{13}$$

Considering the continuity of $Q_0(t)$ at $t = t_r$, we get the following relation from equations (10) and (12):

$$A_2 = A_1 + We^{\tau_{OW}t_d} \tag{14}$$

Again, during the interval $[t_r, T]$, the shortages occur and demand is partially backlogged at the rate $\frac{1}{1+\delta(T-t)}$. The differential equation representing the inventory level is given by

$$\frac{dQ_s}{dt} + B\hat{D} = 0, t_0 \le t \le T \tag{15}$$

Using the boundary condition $Q_s(t_0) = 0$, the solution to equation (15) is obtained as below:

$$Q_s(t) = K(t-t_0) + M\frac{t_0^2 - t^2}{2}, t_0 \le t \le T \tag{16}$$

where

$$K = (\hat{a} - \hat{b}p)(\delta T - 1) + \hat{c}T^2\delta \text{ and } M = (\hat{a} - \hat{b}p)\delta + \hat{c}(\delta T + 1) \tag{17}$$

Putting $t = T$ in equation (16), we get the maximum backordered inventory as below:

$$S_T = -Q_s(T) = -\left[KT - Kt_0 + \frac{M}{2}t_0^2 - \frac{M}{2}T^2\right] \tag{18}$$

Therefore, from equations (6) and (18), the quantity to be ordered per cycle is:

$$\begin{aligned} Q &= S + S_T \\ &= e^{-\tau_{RW}t_d}\left[(\hat{a} - \hat{b}p)(t_r - t_d) + \left\{\tau_{RW}(\hat{a} - \hat{b}p) + \hat{c}\right\}\frac{(t_r^2 - t_d^2)}{2} + \frac{\hat{c}\tau_{RW}}{2}(t_r^3 - t_d^3)\right] \\ &\quad + W + (\hat{a} - \hat{b}p)t_d + \frac{\hat{c}}{2}t_d^2 - KT + Kt_0 - \frac{M}{2}t_0^2 + \frac{M}{2}T^2 \end{aligned} \tag{19}$$

Let $a_1 = a_2 = \hat{a} - \hat{b}p$, $b_1 = \tau_{RW}(\hat{a} - \hat{b}p) + \hat{c}$, $b_2 = \tau_{OW}(\hat{a} - \hat{b}p) + \hat{c}$, $c_1 = \hat{c}\tau_{RW}$, $c_2 = \hat{c}\tau_{OW}$

$$W_1 = W + W\tau_{OW}t_d \text{ and } A_3 = (\hat{a} - \hat{b}p)t_d + \left\{\tau_{RW}(\hat{a} - \hat{b}p) + \hat{c}\right\}\frac{t_d^2}{2} + \frac{\hat{c}\tau_{RW}}{2}t_d^3. \tag{20}$$

Since $\tau_i = \theta_i e^{-\gamma\xi} = \frac{\theta_i}{e^{\gamma\xi}} \le \theta_i << 1$, by using Taylor's expansion and neglecting higher order terms, we get $e^{-\tau_i t} = 1 - \tau_i t$ and $e^{\tau_i t} = 1 + \tau_i t$, where $i = RW, OW$. For simplicity of calculations, these approximations and the notations made in equation (20) will be used to calculate expressions for different costs involved in the system.

The total average cost per cycle has the following elements:

1 Since the replenishment is done at the beginning, ordering cost per cycle is $OC = A$.

2 The inventory holding cost in RW during $[0, T]$ is

$$HC_{RW} = \hat{H}_{RW}\int_0^{t_r} Q_r(t)dt$$
$$= \hat{H}_{RW}\int_0^{t_d} Q_r(t)dt + \hat{H}_{RW}\int_{t_d}^{t_r} Q_r(t)dt$$
$$= \hat{H}_{RW}\Big[-Wt_d - \frac{1}{2}a_1 t_d^2 - \frac{1}{6}\hat{c}t_d^3 + t_d\Big\{W + \big(\hat{a}+\hat{b}p\big)t_d + \frac{1}{2}ct_d^2 + (A_1 - A_3)(1 - t_d\tau_{RW})\Big\}$$
$$-A_1 t_d + \frac{1}{2}a_1 t_d^2 + \frac{1}{6}b_1 t_d^3 + \frac{1}{8}c_1 t_d^4 + A_1 t_r - \frac{1}{2}a_1 t_r^2 - \frac{1}{6}b_1 t_r^3 - \frac{1}{8}c_1 t_r^4 + \frac{1}{2}A_1 t_d^2\tau_{RW}$$
$$-\frac{1}{3}a_1 t_d^3\tau_{RW} - \frac{1}{8}b_1 t_d^4\tau_{RW} - \frac{1}{10}c_1 t_d^5\tau_{RW} - \frac{1}{2}A_1 t_r^2\tau_{RW} + \frac{1}{3}a_1 t_r^3\tau_{RW} + \frac{1}{8}b_1 t_r^4\tau_{RW}$$
$$+\frac{1}{10}c_1 t_r^5\tau_{RW}\Big]$$

3 The inventory holding cost in OW during [0, *T*] is

$$HC_{OW} = \hat{H}_{OW}\int_0^{t_0} Q_r(t)dt$$
$$= \hat{H}_{OW}\int_0^{t_d} Q_0(t)dt + \hat{H}_{OW}\int_{t_d}^{t_r} Q_0(t)dt + \hat{H}_{OW}\int_{t_r}^{t_0} Q_0(t)dt$$
$$= \hat{H}_{OW}\Big[-t_d W_1 + t_r W_1 + \frac{1}{2}Wt_d^2\tau_{OW} - \frac{1}{2}Wt_r^2\tau_{OW} + A_2 t_0 - \frac{1}{2}a_2 t_0^2 - \frac{1}{6}b_2 t_0^3$$
$$-\frac{1}{8}c_2 t_0^4 - A_2 t_r + \frac{1}{2}a_2 t_r^2 + \frac{1}{6}b_2 t_r^3 + \frac{1}{8}c_2 t_r^4 - \frac{1}{2}A_2 t_0^2\tau_{OW} + \frac{1}{3}a_2 t_0^3\tau_{OW} + \frac{1}{8}b_2 t_0^4\tau_{OW}$$
$$+\frac{1}{10}c_2 t_0^5\tau_{OW} + \frac{1}{2}A_2 t_r^2\tau_{OW} - \frac{1}{3}a_2 t_r^3\tau_{OW} - \frac{1}{8}b_2 t_r^4\tau_{OW} - \frac{1}{10}c_2 t_r^5\tau_{OW} + W_{td}\Big]$$

4 The deterioration cost in RW during [0, *T*] is

$$DC_{RW} = \hat{d}_{RW}\int_{t_d}^{t_r} \tau_{RW} Q_r(t)dt$$
$$= \hat{d}_{RW}\tau_{RW}\Big[-A_1 t_d + \frac{1}{2}a_1 t_d^2 + \frac{1}{6}b_1 t_d^3 + \frac{1}{8}c_1 t_d^4 + A_1 t_r - \frac{1}{2}a_1 t_r^2 - \frac{1}{6}b_1 t_r^3 - \frac{1}{8}c_1 t_r^4$$
$$+\frac{1}{2}A_1 t_d^2\tau_{RW} - \frac{1}{3}a_1 t_d^3\tau_{RW} - \frac{1}{8}b_1 t_d^4\tau_{RW} - \frac{1}{10}c_1 t_d^5\tau_{RW} - \frac{1}{2}A_1 t_r^2\tau_{RW} + \frac{1}{3}a_1 t_r^3\tau_{RW}$$
$$+\frac{1}{8}b_1 t_r^4\tau_{RW} + \frac{1}{10}c_1 t_r^5\tau_{RW}\Big]$$

5 The deterioration cost in OW during [0, *T*] is

$$DC_{OW} = \hat{d}_{OW} \int_{t_d}^{t_0} \tau_{OW} Q_0(t)dt$$

$$= \hat{d}_{OW} \int_{t_d}^{t_r} \tau_{OW} Q_0(t)dt + \hat{d}_{OW} \int_{t_r}^{t_0} \tau_{OW} Q_0(t)dt$$

$$= \hat{d}_{OW} \tau_{OW} \Big[-t_d W_1 + t_r W_1 + \frac{1}{2} W t_d^2 \tau_{OW} - \frac{1}{2} W t_r^2 \tau_{OW} + A_2 t_0 - \frac{1}{2} a_2 t_0^2 - \frac{1}{6} b_2 t_0^3$$
$$- \frac{1}{8} c_2 t_0^4 - A_2 t_r + \frac{1}{2} a_2 t_r^2 + \frac{1}{6} b_2 t_r^3 + \frac{1}{8} c_2 t_r^4 - \frac{1}{2} A_2 t_0^2 \tau_{OW} + \frac{1}{3} a_2 t_0^3 \tau_{OW}$$
$$+ \frac{1}{8} b_2 t_0^4 \tau_{OW} + \frac{1}{10} c_2 t_0^5 \tau_{OW} + \frac{1}{2} A_2 t_r^2 \tau_{OW} - \frac{1}{3} a_2 t_r^3 \tau_{OW} - \frac{1}{8} b_2 t_r^4 \tau_{OW} - \frac{1}{10} c_2 t_r^5 \tau_{OW} \Big]$$

6 The shortage cost involved in the system is

$$SC = -\hat{S}_c \int_{t_0}^{T} Q_s(t)dt$$

$$= -\hat{S}_c \left[\frac{K}{2}\left(T^2 - t_0^2\right) - K t_0 \left(T - t_0\right) + \frac{M}{2} t_0^2 \left(T - t_0\right) - \frac{M}{6}\left(T^3 - t_0^3\right) \right]$$

7 The lost sale cost due to partial backlogging is

$$LSC = \hat{L}_c \int_{t_0}^{T} [1 - B(t)] D dt$$

$$= \hat{L}_c \left(\hat{a}T - \hat{b}pT + \frac{\hat{c}}{2} T^2 - \hat{a} t_0 + \hat{b} p t_0 - \frac{\hat{c}}{2} t_0^2 \right) + \hat{L}_c \left[K\left(T - t_0\right) + \frac{M}{2}\left(t_0^2 - T^2\right) \right]$$

8 The purchase cost is

$$PC = xQ$$

$$= x\left[W + \left(\hat{a} - \hat{b}p\right) t_d + \frac{\hat{c}}{2} t_d^2 + \left(1 - \tau_{RW} t_d\right)\left(A_1 - A_3\right) - KT + K t_0 - \frac{M}{2} t_0^2 + \frac{M}{2} T^2 \right]$$

9 The preservation technology investment is

$$PTI = T\xi$$

Thus, adding all these costs calculated above, we get the total average cost per cycle as

$$TAC=\left(p,t_r,t_0,T,\xi\right)=\frac{1}{T}[\text{ordering cost}+\text{holding cost in OW}$$
$$+\text{deterioration cost in RW}+\text{deterioration cost in OW}+\text{shortage cost}$$
$$+\text{lost sale cost}+\text{purchase cost}+\text{preservation technology investment}]$$
$$=\frac{1}{T}[OC+HC_{RW}+HC_{OW}+DC_{RW}+DC_{RW}+SC+LSC+PC+PTI]$$
$$=\frac{1}{T}\Bigg[A+\hat{d}_{RW}\tau_{RW}\Bigg[-A_1t_d+\frac{1}{2}a_1t_d^2+\frac{1}{6}b_1t_d^3+\frac{1}{8}c_1t_d^4+A_1t_r-\frac{1}{2}a_1t_r^2-\frac{1}{6}b_1t_r^3-\frac{1}{8}c_1t_r^4$$
$$+\frac{1}{2}A_1t_d^2\tau_{RW}-\frac{1}{3}a_1t_d^3\tau_{RW}-\frac{1}{8}b_1t_d^4\tau_{RW}-\frac{1}{10}c_1t_d^5\tau_{RW}-\frac{1}{2}A_1t_r^2\tau_{RW}+\frac{1}{3}a_1t_r^3\tau_{RW}$$
$$+\frac{1}{8}b_1t_r^4\tau_{RW}+\frac{1}{10}c_1t_r^5\tau_{RW}\Bigg]+\hat{d}_{OW}\tau_{OW}\Bigg[-t_dW_1+t_rW_1+\frac{1}{2}Wt_d^2\tau_{OW}-\frac{1}{2}Wt_r^2\tau_{OW}$$
$$+A_2t_0-\frac{1}{8}c_2t_0^4-\frac{1}{2}a_2t_0^2-\frac{1}{6}b_2t_0^3-\frac{1}{8}c_2t_0^4-A_2t_r+\frac{1}{2}a_2t_r^2+\frac{1}{6}b_2t_r^3+\frac{1}{8}c_2t_r^4$$
$$-\frac{1}{2}A_2t_0^2\tau_{OW}+\frac{1}{3}a_2t_0^3\tau_{OW}+\frac{1}{8}b_2t_0^4\tau_{OW}+\frac{1}{10}c_2t_0^5\tau_{OW}+\frac{1}{2}A_2t_r^2\tau_{OW}-\frac{1}{3}a_2t_r^3\tau_{OW}$$
$$-\frac{1}{8}b_2t_r^4\tau_{OW}-\frac{1}{10}c_2t_r^5\tau_{OW}\Bigg]+\hat{H}_{RW}\Bigg[-Wt_d-\frac{1}{2}a_1t_d^2-\frac{1}{6}\hat{c}t_d^3+t_d\Bigg\{W+\left(\hat{a}+\hat{b}p\right)t_d$$
$$+\frac{1}{2}ct_d^2+(A_1-A_3)(1-t_d\tau_{RW})\Bigg\}-A_1t_d+\frac{1}{2}a_1t_d^2+\frac{1}{6}b_1t_d^3+\frac{1}{8}c_1t_d^4+A_1t_r-\frac{1}{2}a_1t_r^2$$
$$-\frac{1}{6}b_1t_r^3-\frac{1}{8}c_1t_r^4+\frac{1}{2}A_1t_d^2\tau_{RW}-\frac{1}{3}a_1t_d^3\tau_{RW}-\frac{1}{8}b_1t_d^4\tau_{RW}-\frac{1}{10}c_1t_d^5\tau_{RW}-\frac{1}{2}A_1t_r^2\tau_{RW}$$
$$+\frac{1}{3}a_1t_r^3\tau_{RW}+\frac{1}{8}b_1t_r^4\tau_{RW}+\frac{1}{10}c_1t_r^5\tau_{RW}\Bigg]+\hat{H}_{OW}\Bigg[-t_dW_1+t_rW_1+\frac{1}{2}Wt_d^2\tau_{OW}$$
$$-\frac{1}{2}Wt_r^2\tau_{OW}+A_2t_0-\frac{1}{2}a_2t_0^2-\frac{1}{6}b_2t_0^3-\frac{1}{8}c_2t_0^4-A_2t_r+\frac{1}{2}a_2t_r^2+\frac{1}{6}b_2t_r^3+\frac{1}{8}c_2t_r^4$$
$$-\frac{1}{2}A_2t_0^2\tau_{OW}+\frac{1}{3}a_2t_0^3\tau_{OW}+\frac{1}{8}b_2t_0^4\tau_{OW}+\frac{1}{10}c_2t_0^5\tau_{OW}+\frac{1}{2}A_2t_r^2\tau_{OW}-\frac{1}{3}a_2t_r^3\tau_{OW}$$
$$-\frac{1}{8}b_2t_r^4\tau_{OW}-\frac{1}{10}c_2t_r^5\tau_{OW}+W_{td}\Bigg]-\hat{S}_c\Bigg[\frac{K}{2}\left(T^2-t_0^2\right)-Kt_0\left(T-t_0\right)$$
$$+\frac{M}{2}t_0^2\left(T-t_0\right)-\frac{M}{6}\left(T^3-t_0^3\right)\Bigg]\hat{L}_c\left(\hat{a}T-\hat{b}pT+\frac{\hat{c}}{2}T^2-\hat{a}t_0+\hat{b}pt_0-\frac{\hat{c}}{2}t_0^2\right)$$
$$+\hat{L}_c\Bigg[K\left(T-t_0\right)+\frac{M}{2}\left(t_0^2-T^2\right)\Bigg]+x\Bigg[W+\left(\hat{a}-\hat{b}p\right)t_d+\frac{\hat{c}}{2}t_d^2$$
$$+(1-\tau_{RW}t_d)(A_1-A_3)-KT+Kt_0-\frac{M}{2}t_0^2+\frac{M}{2}T^2\Bigg]+T\xi\Bigg]$$

Our objective is to minimise *TAC*, which is a function of five variables. The necessary condition for optimal solution is

$$\frac{\partial TAC}{\partial p}=0,\frac{\partial TAC}{\partial t_r}=0,\frac{\partial TAC}{\partial t_0}=0,\frac{\partial TAC}{\partial T}=0\text{ and }\frac{\partial TAC}{\partial \xi}=0 \tag{21}$$

The values of p, t_r, t_0, T and ξ obtained by solving equation (21) will be optimal if the Hessian matrix H_{es} given below is positive definite.

$$H_{es} = \begin{bmatrix} \frac{\partial^2 TAC}{\partial p^2} & \frac{\partial^2 TAC}{\partial p \partial t_r} & \frac{\partial^2 TAC}{\partial p \partial t_0} & \frac{\partial^2 TAC}{\partial p \partial T} & \frac{\partial^2 TAC}{\partial p \partial \xi} \\ \frac{\partial TAC}{\partial t_r \partial p} & \frac{\partial^2 TAC}{\partial t_r^2} & \frac{\partial TAC}{\partial t_r \partial t_0} & \frac{\partial^2 TAC}{\partial t_r \partial T} & \frac{\partial^2 TAC}{\partial t_r \partial \xi} \\ \frac{\partial^2 TAC}{\partial t_0 \partial p} & \frac{\partial^2 TAC}{\partial t_0 \partial t_r} & \frac{\partial^2 TAC}{\partial t_0^2} & \frac{\partial^2 TAC}{\partial t_0 \partial T} & \frac{\partial^2 TAC}{\partial t_0 \partial \xi} \\ \frac{\partial^2 TAC}{\partial T \partial p} & \frac{\partial^2 TAC}{\partial T \partial t_r} & \frac{\partial^2 TAC}{\partial T \partial t_0} & \frac{\partial^2 TAC}{\partial T^2} & \frac{\partial^2 TAC}{\partial T \partial \xi} \\ \frac{\partial^2 TAC}{\partial \xi \partial p} & \frac{\partial^2 TAC}{\partial \xi \partial t_r} & \frac{\partial^2 TAC}{\partial \xi \partial t_0} & \frac{\partial^2 TAC}{\partial \xi \partial T} & \frac{\partial^2 TAC}{\partial \xi^2} \end{bmatrix}.$$

6 Solution procedure

In this work, parametric functional form representation (see Appendix 1) of interval number is used to represent the imprecise parameters. Interval parameters have been represented in parametric form as an expression and used to find the total average cost of the system. Normally, using the interval, two separate objective functions are formulated corresponding to lower and upper limits of an interval whose optimisation requires heuristic or meta-heuristic search methods. Here, the parametric approach is used to get the closed form of expression of the objective function, which leads to a single objective optimisation problem, and hence can be solved by exiting classical or numerical methods. This approach of parametric functional form has been used in Das and Roy (2018). As the objective function in our model is highly nonlinear in nature, it is not possible to solve it analytically. So, we have solved the model numerically. The following steps are adopted to get the optimal values of p, t_r, t_0, T, ξ, Q and TAC.

Algorithm

Step1 Set the parameters: A, t_d, W, x, δ, γ, θ_{RW} and θ_{OW}.

Step 2 Set the imprecise values of $\hat{a}, \hat{b}, \hat{c}, \hat{H}_{RW}, \hat{H}_{OW}, \hat{d}_{RW}, \hat{d}_{OW}, \hat{S}_c$ and $\hat{L}_c$ in terms of interval numbers.

Step 3 Find the values of $a(m)$, $b(m)$, $c(m)$, $H_{RW}(m)$, $H_{OW}(m)$, $d_{RW}(m)$, $d_{OW}(m)$, $S_c(m)$ and $L_c(m)$ using parametric functional form for a particular value of m.

Step 4 Evaluate the total average cost function TAC.

Step 5 Find the optimal values of the decision variables.

Step 6 Calculate the optimal values Q^* and TAC^*.

7 Numerical illustration

In this section, a numerical example is provided for illustrating the model. The following values of parameter are taken as input:

$$A = \text{Rs}1000,\ t_d = 3/12 = 0.25 \text{ yr.},\ W = 200 \text{ units},\ x = \text{Rs}70 \text{ per unit},\ \delta = 0.02,$$
$$\gamma = 0.3,\ \theta_{RW} = 0.015 \text{ and } \theta_{OW} = 0.5.$$

The imprecise values of inventory parameters represented by interval numbers are as follows:

$$\hat{a} = [1000, 1050],\ \hat{b} = [2, 3],\ \hat{c} = [1, 2],\ \hat{L}_c = [4, 5],\ \hat{S}_c = [2, 3],\ \hat{d}_{RW} = [1.5, 2],$$
$$\hat{d}_{OW} = [2, 4],\ \hat{H}_{RW} = [4, 6],\ \hat{H}_{OW} = [2, 3].$$

Using the parametric functional form for $m = 0.5$, we have,

$$a(m) = 1024.7,\ b(m) = 2.44949,\ c(m) = 1.41421,\ H_{RW}(m) = 4.89898,$$
$$H_{OW}(m) = 2.44949,\ d_{RW}(m) = 1.73205,\ d_{OW}(m) = 2.82843,\ L_c(m) = 4.47214,$$
$$S_c(m) = 2.44949.$$

The optimal solution obtained is as below:

$$p^* = \text{Rs}199.516 \approx \text{Rs}200,\ t_r^* = 0.41667 \text{ yr.},\ t_0^* = 2.13094 \text{ yr.},\ T^* = 2.46427 \text{ yr.},$$
$$\xi^* = \text{Rs}10.$$

And the optimal values of the ordering quantity and total average cost are respectively

$$Q^* = 602.865 \approx 603 \text{ units},\ TAC = \text{Rs}21599.1 \approx \text{Rs}21600.$$

The above optimal solution is obtained for $m = 0.5$. The reason for taking $m = 0.5$ is that corresponding to $m = 0.5$, we get the least total average cost among the optimal solutions (presented in Table 2) obtained for different values of m. Moreover, in Table 2, along with the optimal solutions, we represent the approximate values of $a(m)$, $b(m)$, $c(m)$, $H_{RW}(m)$, $H_{OW}(m)$, $d_{RW}(m)$, $d_{OW}(m)$, $S_c(m)$ and $L_c(m)$ using parametric functional form for different values of m. The benefit of varying m from 0 to 1 is that we use lower, intermediate and upper bounds of the interval valued parameters to obtain the corresponding optimal solutions of the model. It is observed from Table 2 that when m increases from 0 to 0.4, there are some fluctuations in the optimal values TAC^*. The same trend is also observed in case of t_0^*, T^*, p^* and Q^*. But when m moves from 0.5 to 1, TAC^* as well as Q^* increases. Whereas the values of t_0^*, T^* and p^* decrease when m moves from 0.5 to 1. Some fluctuations in the values of t_r^* and ξ^* are observed when m moves from 0 to 1. Graphical representations of the optimal values with respect to different values of m are given in Appendix 2. In the next section, we perform sensitivity analysis of the optimal solution obtained for $m = 0.5$.

Table 2 Optimal solutions obtained for different values of m

m	$a(m)$	$b(m)$	$c(m)$	$H_{RW}(m)$	$H_{OW}(m)$	$d_{RW}(m)$	$d_{OW}(m)$	$s_c(m)$	$L_c(m)$	p^*	t_r^*	t_0^*	T^*	ξ^*	Q^*	TAC^*
0	1,000	2	1	4	2	1.5	2	2	4	180.728	0.41667	2.22694	2.56027	10	679.404	23,358.2
0.1	1,004.89	2.08276	1.07177	4.16552	2.08276	1.54378	2.14355	2.08276	4.09026	106.496	0.41667	1.66331	1.99665	11.5537	787.753	33,534.2
0.2	1,009.81	2.16894	1.1487	4.33789	2.16894	1.58884	2.2974	2.16894	4.18256	108.771	0.41667	2.13284	2.46617	10	780.925	27,730.3
0.3	1,014.74	2.25869	1.23114	4.51739	2.25869	1.63521	2.46229	2.25869	4.27694	129.414	0.41667	2.11576	2.44909	10	742.432	26,544.2
0.4	1,019.71	2.35216	1.31951	4.70432	2.35216	1.68293	2.63902	2.35216	4.37345	82.3931	1.26686	1.67993	2.01323	10.1209	1538.86	57,069.4
0.5	*1,024.7*	*2.44949*	*1.41421*	*4.89898*	*2.44949*	*1.73205*	*2.82843*	*2.44949*	*4.47214*	*199.516*	*0.41667*	*2.13094*	*2.46427*	*10*	*602.865*	*21,599.1*
0.6	1,029.71	2.55085	1.51572	5.1017	2.55085	1.7826	3.03143	2.55085	4.57305	154.39	0.41667	2.0564	2.38974	10	677.784	24,824.8
0.7	1,034.74	2.6564	1.6245	5.3128	2.6564	1.83463	3.24901	2.6564	4.67624	149.418	0.41767	2.03246	2.36579	10.001	679.956	25,170.7
0.8	1,039.8	2.76632	1.7411	5.53265	2.76632	1.88818	3.4822	2.76632	4.78176	144.983	0.41667	1.99927	2.33261	10	680.077	25,446.1
0.9	1,044.89	2.88079	1.86607	5.76159	2.88079	1.94328	3.73213	2.88079	4.88966	118.57	0.41767	1.96093	2.29426	10.001	729.25	27,726.5
1	1,050	3	2	6	3	2	4	3	5	96.7013	0.41766	1.92252	2.25537	10.001	771.422	29,769.4

8 Sensitivity analysis

In this section, to study the effect of changes in the parameters values on the optimal values, we perform sensitivity analysis. These analyses have been carried out by changing one parameter by ±5% and ±10% at a time and keeping other parameters at their original values. The results of these analyses are shown in Table 3 to Table 8.

Table 3 Sensitivity table for A

	A	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	1,100	0.416667	2.05231	2.38564	10	108.988	769.749	28,194.9
5%	1,050	0.416667	2.06515	2.39848	10	133.766	724.229	26,387.5
0%	1,000	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	950	0.468247	3.07203	3.43032	10.0101	264.386	513.318	21,040.5
–10%	900	0.577903	1.92417	2.77241	10.8231	274.194	704.846	20,511.1

Table 4 Sensitivity table for θ_{RW}

	θ_{RW}	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	0.0165	0.416668	2.07175	2.40509	10	134.395	722.418	26,321.5
5%	0.01575	0.416667	2.07578	2.40911	10	133.573	723.904	26,380.7
0%	0.015	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	0.01425	0.417579	2.1081	2.44053	10.001	181.679	635.061	22,907.6
–10%	0.0135	0.416667	2.07467	2.408	10	148.643	696.19	25,291.3

Table 5 Sensitivity table for θ_{OW}

	θ_{OW}	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	0.55	0.496701	2.10664	2.71841	8.9765	112.269	1,031.11	30,068.2
5%	0.525	0.496923	1.88601	2.36723	10.3177	159.976	819.918	28,827.3
0%	0.5	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	0.475	0.518639	1.76474	2.55244	10.3144	275.612	657.583	19,588
–10%	0.45	0.54142	1.83367	2.61667	11.1094	285.542	632.094	18,941.7

Table 6 Sensitivity table for x

	x	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	77	0.417602	2.16311	2.49562	10.001	191.148	618.309	23,970.4
5%	73.5	0.416667	2.07025	2.40358	10	91.7801	800.6	30,562.6
0%	70	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	66.5	0.470724	0.944707	1.2765	11.9442	67.0482	891.435	48,569.6
–10%	63	0.416667	2.02123	2.35456	10	142.334	707.779	23,671.3

Table 7 Sensitivity table for δ

	δ	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	0.022	0.416667	1.54859	1.88192	12	141.172	709.798	32,717.9
5%	0.021	0.416667	2.10171	2.43504	10	167.626	661.356	23,916.4
0%	0.02	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	0.019	0.416667	2.05904	2.39237	10	107.125	772.47	28,288.4
–10%	0.018	0.416667	1.54867	1.882	12	141.196	709.906	32,720.9

Table 8 Sensitivity table for W

	W	t_r^*	t_0^*	T^*	ξ^*	p^*	Q^*	TAC^*
10%	220	0.417651	2.07349	2.4059	10.001	110.005	787.203	28,631.5
5%	210	0.416667	2.07232	2.40565	10	130.264	739.959	26,886.4
0%	200	0.41667	2.13094	2.46427	10	199.516	602.865	21,599.1
–5%	190	0.431673	2.04731	2.41747	10.1435	251.925	517.826	18,211.5
–10%	180	0.436465	1.92869	2.26208	10.8963	279.923	442.009	17,121.7

The following observations can be obtained from Table 3 to Table 8:

- TAC^* increases with the increase in the parameters A which is quite apparent. At first, it increases gradually (for –10% to –0% change in A) then increases rapidly (for 0% to 10% change in A). TAC^* increases sharply with the increase in the parameter θ_{OW}, which is fairly usual occurrence. Similar high increase in the values of TAC^* is noticed with the increase of the parameter W. When the values of parameters θ_{RW}, δ and x increase, TAC^* fluctuates very significantly. This indicates that TAC^* is highly sensitive with regard to all the parameters A, θ_{RW}, θ_{OW}, x, δ and W.

- Q^* shows high rise in its values when W increases. Very significant variations in the values of Q^* are also observed for the changes in the parameters θ_{OW} and x. However, Q^* changes reasonably with the increase in A, θ_{RW} and δ. This means that Q^* is highly sensitive with respect to θ_{OW}, x and W; moderately sensitive with respect to A, θ_{RW} and δ.

- Similar analysis shows that the change in t_r^* with the increase in θ_{OW} is significant. When A changes from –10% to 0%, the value of t_r^* decrease reasonably but its value remains unchanged corresponding to 5% and 10% change in A. Moreover, t_r^* changes slightly with the changes in x and W, and it remains almost static with the changes in θ_{RW} and δ. This specifies that t_r is moderately sensitive with respect to A and θ_{OW}; less sensitive with respect to x and W; very less sensitive with respect to θ_{RW} and δ.

- ξ^* shows some fluctuations with the increase in the values of θ_{OW} and δ increase. But, when θ_{RW} increases, ξ^* remains almost constant. Moreover, slight fluctuations in the values of ξ^* are observed with changes in the values of the parameters A, x and W. This indicates that ξ^* is moderately sensitive with regard to θ_{OW} and δ; less sensitive with regard to A, x and W; very less sensitive with regard to θ_{RW}.

- The shortage occurrence time (t_0^*) is highly sensitive to the changes in A, θ_{OW}, x, δ and W, it is moderately sensitive to the change in θ_{RW}. Similar trends are also observed for T^*. That is, the replenishment time T^* is highly sensitive with respect to A, θ_{OW}, x, δ and W; it is moderately sensitive with respect to θ_{RW}.
- The effects of increase in the values of A, θ_{OW} and W on p^* are very significant. p^* decreases rapidly with the increase in the values of the mentioned parameters. Noticeable variations are also observed with the increase in the values of θ_{RW}, δ and x. Thus, similar to TAC^*, p^* is also highly sensitive with regard to all the parameters A, θ_{RW}, θ_{OW}, x, δ and W.

9 Conclusions

This paper considers a two-warehouse EOQ model for non-instantaneous deteriorating items with price and time dependent demand under interval environment. The coefficients of demand and all inventory cost parameters are considered as interval numbers to represent the practical market situation as these are not always fixed in real life. Shortages have been allowed in the model to make a more realistic scenario, and are partially backlogged at the variable rate dependent on the length of waiting time to the next replenishment. Further, in order to reduce the deterioration rate, preservation technology investment is made in both the warehouses (RW and OW). In the present work, the concept of parametric functional form representation of interval numbers is used to find out total average cost of the model. The advantage of using parametric form is that we get a closed form of expression of the objective function, and can be solved by existing classical or numerical method. To authors' best knowledge, this type of two-warehouse inventory model for non-instantaneous deteriorating items with price and time dependent demand, preservation technology under interval environment has not been reported yet in inventory literature. This work is very helpful to the retailers for decision making in the real market situations where the uncertainty in inventory cost parameters persists. The aim of this present study is to determine retailer's optimal ordering policy so that present worth of the total average cost is minimised. The corresponding two-warehouse inventory model has been developed, and solved numerically due the high nonlinear nature of the objective function. Further, sensitivity analysis (in Section 8) is carried out to study the effects of different parameters on the optimal solution. Based on the analysis, we obtain some useful managerial insights that are highlighted through the following points:

- As the increase in ordering cost increases the total cost per unit time significantly, it is advisable for the decision maker to make decision in lowering investment towards ordering cost. That is, if we curtail the ordering cost, then we may have significant amount of saving, and the total cost per unit time is automatically minimised. So, from practical aspect this model is valid.
- As the unit purchasing cost has significant affects not only on total cost per unit time but also on ordering quantity (as observed form Table 6), it is advisable for the decision maker to perform sufficient analysis on unit purchase cost before buying items.

- As the capacity of OW decreases, total cost per unit time as well as ordering quantity decreases while the selling price increases (as observed from Table 8). Thus, the inventory manager should be encouraged to have OW of lower capacity in order to reduce the total cost per unit time. Further, organisation has an incentive to increase the selling price.

Moreover, this article considers that items are received as soon as it is ordered that is, the lead time is zero. But, in many cases, we observe a delay in receiving items due unavoidable circumstances that could actually make impact in decision making. Again, in this model, transportation cost for transporting items from RW to OW is neglected. But in practice, extra costs like labour charges, vehicle fares, etc., while transporting commodities from RW to OW, are generally incurred in the system. So, one may contemplate and study upon these limitations of this paper. Also, for further research, this present model may be enhanced in different ways. One such way is to incorporate trade credit (single level or two-level) policy. Another extension of this model can be done by considering advertisement factor in demand function. Again, one may also think of studying this model in fuzzy or stochastic environment.

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Appendix 1

This appendix includes definitions of interval number and its parametric functional form representation. These definitions have been taken from Das and Roy (2018).

Definition 1

We define an interval number $\hat{A}$ as a closed interval $[aL, aR] = \{x \in \Re: aL \leq x \leq aR\}$, where aL, aR are lower and upper bounds respectively and $\Re$ is the set of real numbers.

To represent an interval number, we define the corresponding interval function as below.

Definition 2

Let $\hat{A} = [aL, aR]$ be an interval number, where $aR \geq aL > 0$. Then, interval function corresponding to the interval number $\hat{A}$ is nothing but a real valued function f: $[0, 1] \to \Re$ defined by $f(m) = (aL)^{1-m}(aR)^m$ for all $m \in [0, 1]$. It is easy to see that $f(m)$ is a monotone increasing function. Clearly, $f(0) = aL$, $f(1) = aR$. This implies that $f(m) \in [aL, aR]$.

For convenience, we denote interval function corresponding to interval number $\hat{A} = [aL, aR]$ as $A(m)$ that is, $A(m) = f(m) = (aL)^{1-m}(aR)^m$. This representation of interval number as a real number is called parametric functional form representation.

For the following interval numbers:

$$a = [aL, aR], b = [bL, bR], c = [cL, cR], \hat{H}_{RW} = [h_1L, h_1R], \hat{H}_{OW} = [h_2L, h_2R],$$
$$\hat{d}_{RW} = [d_1L, d_1R], \hat{d}_{OW} = [d_2L, d_2R], \hat{S}_c = [S_cL, S_cR], \hat{L}_c = [L_cL, L_cR]$$

The corresponding parametric functional form representations are as below:

$$a(m) = (aL)^{1-m}(aR)^m, b(m) = (bL)^{1-m}(bR)^m, c(m) = (cL)^{1-m}(cR)^m,$$
$$H_{RW}(m) = (h_1L)^{1-m}(h_1R)^m, H_{OW}(m) = (h_2L)^{1-m}(h_2R)^m,$$
$$d_{RW}(m) = (d_1L)^{1-m}(d_1R)^m, d_{OW}(m) = (d_2L)^{1-m}(d_2R)^m,$$
$$S_c(m) = (S_cL)^{1-m}(S_cR)^m, L_c(m) = (L_cL)^{1-m}(L_cR)^m.$$

Appendix 2

This appendix includes graphs showing variations of optimal values TAC^*, T^*, ξ^*, t_r^*, t_0^*, Q^* and p^* with m.

Figure 2 Plot of TAC^* for different values of m (see online version for colours)

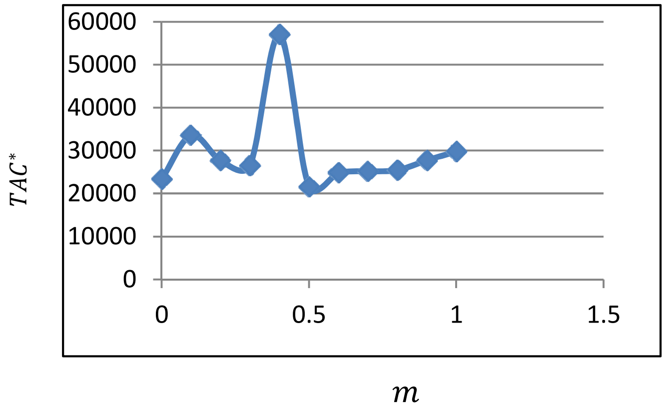

Figure 3 Plot of T^* for different values of m (see online version for colours)

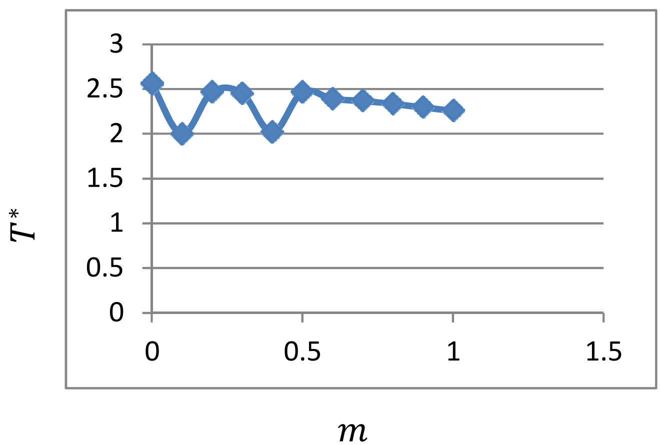

Figure 4 Plot of ξ^* for different values of m (see online version for colours)

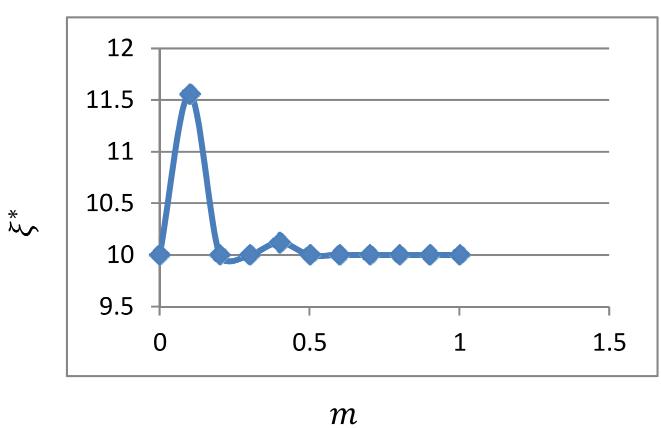

Figure 5 Plot of t_r^* for different values of m (see online version for colours)

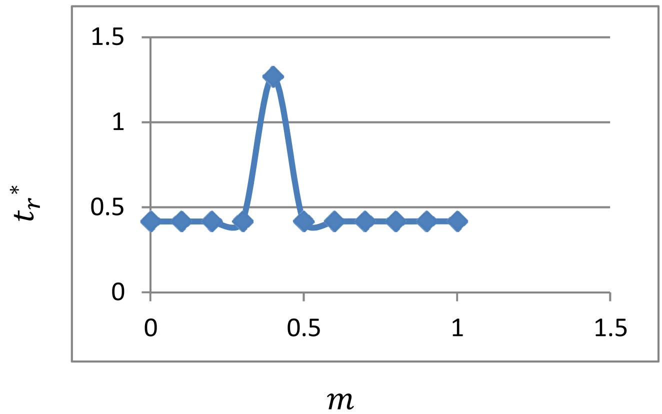

Figure 6 Plot of t_0^* for different values of m (see online version for colours)

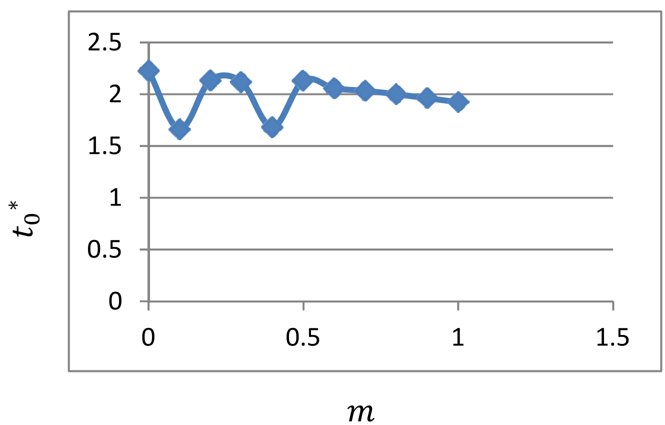

Figure 7 Plot of Q^* for different values of m (see online version for colours)

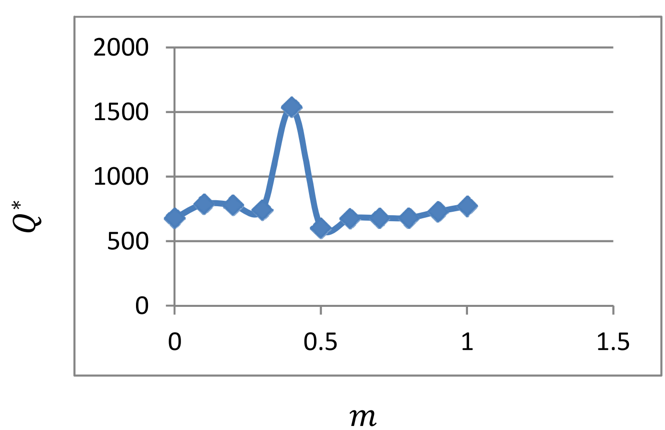

Figure 8 Plot of p^* for different values of m (see online version for colours)